XXII.

CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY.

THE DYNAMIC ACTION OF AN ELECTRIC CURRENT.

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In the measurement of the strength of an electric current by a galvanometer, the formula usually employed is

$$C = \frac{H r}{2 \pi n} \tan \theta.$$

This formula is strictly true only when the needle is upon the axis of the coil, which must consist of a small number of turns of wire. If the needle is not upon the axis of the coil, this formula no longer holds good, and a new one must be determined by the use of elliptic integrals or by zonal harmonics. The method by zonal harmonics is the simpler of the two, but is rendered very difficult by Maxwell in his Treatise on Electricity and Magnetism. Maxwell also devised a method of approximation to find the potential at any point, when a coil of large cross-section was employed. His treatment of this is almost incomprehensible. It is hoped that in the following paper the errors in Maxwell's formulæ have been eliminated, and that the subject has been brought within the grasp of all.

Let us first take a current of electricity passing through a single turn of wire in the form of a circle, and find the potential due to this current at any point. Since, by Ampère's theorem, the magnetic action of a closed current is equal to that of a magnetic shell of the same contour, we will for simplicity replace the circuit by that part of a spherical shell which has the circuit

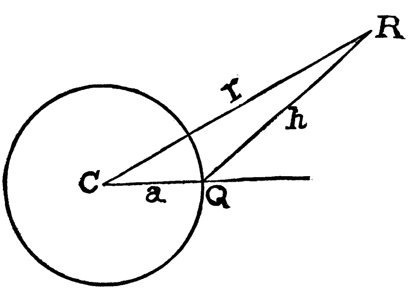

Fig. 1.

for a boundary. Let, now, the surface density of such a spherical magnetic shell be ϕ, its radius be a, and let there be a unit magnetic pole R at distance r from C, the centre of the sphere. Moreover, let dS be any small element at Q, and at a distance h from R. Let $p = \frac{1}{h}$. Let O be the value of the flow, or the surface integral, at the point Q. Then the mutual potential of the magnetic shell and a unit pole placed at R will be

$$\Omega = \int \frac{d\,O}{d\,a}.$$

But, since h is the distance between the element and the point R,

$$O = \int \frac{\phi}{h} d\,S = \int p \phi \, d\,S;$$

whence by substitution we find

$$\Omega = \int\int \phi \frac{dp}{da} d\,S.$$

But p is a homogeneous function of the degree -1 of the radius a and of the distance r from R, which gives the conditions,

$$a \frac{d\,p}{d\,a} + r \frac{d\,p}{d\,r} = -p\,;$$

or,

$$\frac{d\,p}{d\,a} = -\frac{1}{a}\left(p + r\frac{d\,p}{d\,r}\right) = -\frac{1}{a}\left(\frac{d\,(p\,r)}{d\,r}\right);$$

substituting, we have

$$\Omega = -\int\int \frac{\phi}{a} \frac{d\,(p\,r)}{d\,r} d\,S.$$

But if V is the potential at R due to the small element of the shell dS, we shall have

$$V = \int\int \phi p \, d\,S;$$

therefore

$$\Omega = -\frac{1}{a}\frac{d\,(r\,V)}{d\,r}, \qquad (a.)$$

which is the expression for the potential due to a magnetic shell of unit strength.

We must now find V. Let AB be the coil bounding the magnetic shell, and let CZ be the axis of the coil, and consequently normal to the shell. Let us take a point Z upon the axis, distant z from the centre C. The potential at Z due to any small element dS at P upon the surface of the magnetic shell is

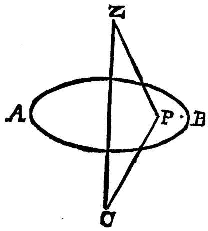

Fig. 2.

$$\frac{dS}{ZP};$$

but

$$ZP = (z^2 + a^2 - 2az\cos\alpha)^{\frac{1}{2}};$$

calling $\cos\alpha = \mu$,

$$\frac{1}{ZP} = (z^2 + a^2 - 2az\mu)^{-\frac{1}{2}},$$

$$\frac{1}{ZP} = \frac{1}{z}\left(1 + \frac{a^2}{z^2} - 2\frac{a}{z}\mu\right)^{-\frac{1}{2}};$$

also,

$$\frac{1}{ZP} = \frac{1}{a}\left(1 + \frac{z^2}{a^2} - 2\frac{z}{a}\mu\right)^{-\frac{1}{2}};$$

calling $\frac{a}{z} = h$ and $\frac{z}{a} = h_1$, we have

$$\frac{1}{ZP} = \frac{1}{z}(1 - 2h\mu + h^2)^{-\frac{1}{2}}, \qquad \text{[Equa. I.]}$$

$$\frac{1}{ZP} = \frac{1}{a}(1 - 2h_1\mu + h_1^2)^{-\frac{1}{2}}. \qquad \text{[Equa. II.]}$$

Developing equation [I.] by the binomial theorem, we have

$$\frac{1}{ZP} = \frac{1}{z}\left[1 + \mu h + \left(\tfrac{12}{8}\mu^2 - \tfrac{1}{2}\right)h^2 + \left(\tfrac{5}{2}\mu^3 - \tfrac{3}{2}\mu\right)h^3 + \&c.\right] \qquad \text{[III.]}$$

$$= \frac{1}{z}(P_0 + P_1 h + P_2 h^2 + P_3 h^3 + \&c.). \qquad \text{[IV.]}$$

Equation IV. is merely a simplified form of equation III., the expressions P_0, P_1, P_2, P_i, &c. being substituted for the coefficients of h^0 or 1, h, h^2, h^i, &c. In the same way equation II. becomes

$$\frac{1}{ZP} = \frac{1}{a}(P_0 + P_1 h_1 + P_2 h_1^2 + P_3 h_1^3 + \&c.). \qquad \text{[V.]}$$

The values of P_0, P_1, P_2, &c. can be readily seen from equation III. Below we write a few values which are determined from [III.].

$$P_0 = 1.$$
$$P_1 = \cos a \text{ or } \mu.$$
$$P_2 = \tfrac{3}{2}\mu^2 - \tfrac{1}{2}.$$
$$P_3 = \tfrac{5}{2}\mu^3 - \tfrac{3}{2}\mu.$$
$$P_4 = \tfrac{7}{2}\cdot\tfrac{5}{4}\mu^4 - \tfrac{5}{2}\cdot\tfrac{3}{2}\mu^2 + \tfrac{3}{2}\cdot\tfrac{1}{4}.$$
$$P_5 = \text{\&c.}$$

Equations [IV.] and [V.] are general expressions for the reciprocal of the distance, or $\frac{1}{ZP}$. We may therefore express

$$\frac{dS}{ZP} = \frac{dS}{z}\left(P_0 + P_1\frac{a}{z} + P_2\frac{a^2}{z^2} + \text{\&c.} + P_i\frac{a^i}{z^i}\right) \qquad \text{[VI.]}$$

$$= \frac{dS}{a}\left(P_0 + P_1\frac{z}{a} + P_2\frac{z^2}{a^2} + \text{\&c.} + P_i\frac{z^i}{a^i}\right). \qquad \text{[VII.]}$$

Equation VI. must be employed if z is greater than a, and [VII.] if z is less.

We made dS a small element of the portion of the spherical shell bounded by the coil.

In polar co-ordinates

$$dS = -a^2 d\mu\, d\phi;$$

S will evidently be the integral of the above when ϕ is integrated between the limits of 0 and 2π, and μ is integrated between $\cos a$ and 1, or

$$S = 2\pi a^2 \int_\mu^1 d\mu.$$

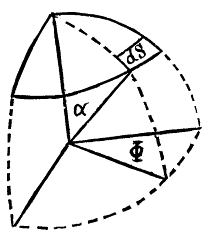

Fig. 3.

Hence the potential at Z due to the included shell is $V = \frac{S}{ZP}$, where ZP is the distance to each element of area of the sphere.

$$V = 2\pi a\left\{\int_\mu^1 P_0\,d\mu + \frac{z}{a}\int_\mu^1 P_1\,d\mu + \text{\&c.} + \frac{z^i}{a^i}\int_\mu^1 P_i\,d\mu\right\}. \qquad \text{[VIII.]}$$

$$V' = 2\pi\frac{a^2}{z}\left\{\int_\mu^1 P_0\,d\mu + \frac{a}{z}\int_\mu^1 P_1\,d\mu + \text{\&c.} + \frac{a^i}{z^i}\int_\mu^1 P_i\,d\mu\right\}. \qquad \text{[IX.]}$$

We found that $P_0 = 1$, hence $\int_\mu^1 P_0 \, d\mu = 1 - \mu$; also $P_1 = \mu$; therefore $\int_\mu^1 \mu \, d\mu$ equals $\frac{1}{2}(1 - \mu^2)$. In the same way, we might find values for all the values of P, but it has been found that all the integrals of the form $\int_\mu^1 P_i \, d\mu$ are satisfied by the equation,

$$\int_\mu^1 P_i \, d\mu = \frac{1 - \mu^2}{i(i+1)} \frac{d P_i}{d\mu}. \qquad \text{[IX. B.]}$$

Substituting this expression in equations VIII. and IX. we have

$$V = 2\pi a \left\{ 1 - \mu + \&c. + \frac{1 - \mu^2}{i(i+1)} \frac{z^i}{a^i} \frac{d P_i}{d\mu} \right\}. \qquad \text{[X.]}$$

$$V' = 2\pi \frac{a^2}{z} \left\{ 1 - \mu + \&c. + \frac{1 - \mu^2}{i(i+1)} \frac{a^i}{z^i} \frac{d P_i}{d\mu} \right\}. \qquad \text{[XI.]}$$

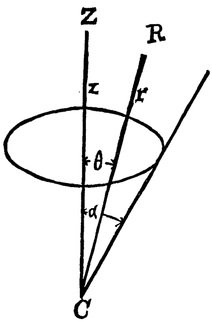

Fig. 4.

These last two equations are the expressions for the potential at any point Z upon the axis. To make these equations general, or for the potential at any point R distant r, it is found that we can substitute r for z, provided each term of equation [X.] and [XI.] be multiplied by the P of the order corresponding to the power of r; the P's being functions of the angle θ which the line CR makes with the axis ZC. We have, then, for the potential at R,

$$V = 2\pi a \left\{ 1 - \cos a + \frac{1 - \cos^2 a}{2} \frac{r}{a} \frac{d P_1(a)}{d \cos a} P_1(\theta) + \&c. \right.$$
$$\left. + \frac{1 - \cos^2 a}{i(i+1)} \frac{r^i}{a^i} \frac{d P_i(a)}{d \cos a} P_i(\theta) \right\}. \qquad \text{[XII.]}$$

$$V' = 2\pi \frac{a^2}{r} \left\{ 1 - \cos a + \frac{1 - \cos^2 a}{2} \frac{a}{r} \frac{d P_1(a)}{d \cos a} P_1(\theta) + \&c. \right.$$
$$\left. + \frac{1 - \cos^2 a}{i(i+1)} \frac{a^i}{r^i} \frac{d P_i(a)}{d \cos a} P_i(\theta) \right\}. \qquad \text{[XIII.]}$$

We can readily see that these expressions are true; for if we make $\theta = 0$ and $r = z$, we shall have equations [X.] and [XI.]. Moreover we find that each of these expressions satisfies Laplace's equation.

Substituting the value of V and V' from equations [XII.] and [XIII.] for V in the expression

$$\Omega = -\frac{1}{a}\frac{d(rV)}{dr},$$

and differentiating, we find

$$\Omega = -2\pi\left\{1 - \cos a + \&c. + \frac{\sin^2 a}{i}\frac{r^i}{a^i}\frac{dP_i(a)}{d\cos a}P_i(\theta)\right\}; \quad [XIV.]$$

$$\Omega' = 2\pi\sin^2 a\left\{\frac{1}{2}\frac{a^2}{r^2}P_1(\theta)\frac{dP_1(a)}{d\cos a} + \&c. + \frac{1}{i+1}\frac{a^{i+1}}{r^{i+1}}\frac{dP_i(a)}{d\cos a}P_i(\theta)\right\}. \quad [XV.]$$

These are the most general expressions for the potential at any point whatever, due to a single coil of wire. Equation [XIV.] must be employed if r is less than a, or if the point lies within the sphere of which a is the radius and [XV.] for all points outside.

If the origin had been taken at the centre of the circular coil, a would have been equal to $\frac{\pi}{2}$ or $90°$. Substituting this value for a in [XIV.] and [XV.],

$$\Omega = -2\pi\left\{1 - \cos 90° + \&c. + \frac{\sin^2 90°}{i}\frac{r^i}{a^i}\frac{dP_i(90°)}{d\cos 90°}P_i(\theta)\right\};$$

$$\Omega' = 2\pi\sin^2 90°\left\{\frac{1}{2}\frac{a^2}{r^2}P_1(\theta)\frac{dP_1(90°)}{d\cos 90°} + \&c. + \frac{1}{i+1}\frac{a^{i+1}}{r^{i+1}}\frac{dP_i(90°)}{d\cos 90°}P_i(\theta)\right\}.$$

Noticing that $\cos 90° = 0$, and that all the even orders of $\frac{dP}{d\cos a}$ are multiplied by $\cos a$, and therefore equal to 0, we have

$$\Omega = -2\pi\left\{1 + \frac{r}{a}P_1(\theta) - \frac{1}{2}\frac{r^3}{a^3}P_3(\theta) + \frac{3}{8}\frac{r^5}{a^5}P_5(\theta) - \frac{5}{16}\frac{r^7}{a^7}P_7(\theta) + \&c.;\right.$$

$$\Omega' = 2\pi\left\{\frac{1}{2}\frac{a^2}{r^2}P_1(\theta) - \frac{3}{8}\frac{a^4}{r^4}P_3(\theta) + \frac{5}{16}\frac{a^6}{r^6}P_5(\theta) - \frac{35}{128}\frac{a^8}{r^8}P_7(\theta) + \&c.\right.$$

We have heretofore supposed the coil to have been made of a single turn of wire. If the coil consists of a number of turns of wire, the potential at any point may be obtained by the following approximation. Let the single turn, the potential due to which at any point we found could be expressed by equation [XIV.] or [XV.], occupy the centre of the coil whose rectangular dimensions are ξ and η. Let the co-ordinates of the wire at the centre of the coil be x and y. Now the potential at O due to the coils whose cross-sectional area is $\xi\eta$ will be a function of x and y. If we look at equation [XIV.],

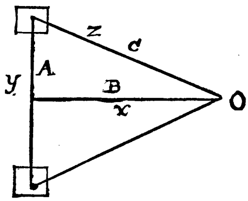

Fig. 5.

$$\Omega = -2\pi + 2\pi\cos a - 2\pi\frac{\sin^2 a}{a}\frac{d\,P_1(a)}{d\cos a}\,r\,P_1(\theta) + \&c.$$

$$+ 2\pi\frac{\sin^2 a}{i\,a^i}\frac{d\,P_i(a)}{d\cos a}\,r^i P_i(\theta), \qquad \text{[XVI. A.]}$$

or, as it may be written,

$$\Omega = -2\pi + 2\pi Q_0 - 2\pi Q_1 r P_1(\theta) + \&c. + 2\pi Q_i r^i P_i(\theta), \quad \text{[XVI. B.]}$$

we see that Q_0, Q_1, Q_i, &c. are such functions of x and y.

Let G be the mean value of Q for the values of Q for each wire within the limits $+\frac{1}{2}\xi$, $-\frac{1}{2}\xi$, $+\frac{1}{2}\eta$, and $-\frac{1}{2}\eta$, or

$$G = \frac{\int_{-\frac{1}{2}\xi}^{+\frac{1}{2}\xi}\int_{-\frac{1}{2}\eta}^{+\frac{1}{2}\eta} Q\,dx\,dy}{\xi\eta}. \qquad \text{[XVII.]}$$

This expression gives the value of a new coefficient G for the coil whose cross-section is $\xi\eta$ in place of Q, the coefficient for a single turn. From this we see that the potential at a point O due to the coil $\xi\eta$ is

$$\Omega = -2\pi + 2\pi G_0 - 2\pi G_1 r P_1(\theta) + \&c. + 2\pi G_i r^i P_i(\theta); \quad \text{[XVIII.]}$$

or, calling $G' = 2\pi G$, we have

$$\Omega = -2\pi + G_0' - G_1' r P_1(\theta) + \&c. + G_i' r^i P_i(\theta). \qquad \text{[XIX.]}$$

Expanding the numerator of equation [XVII.] by Taylor's theorem, integrating between the limits $\frac{1}{2}\xi$, $-\frac{1}{2}\xi$, $\frac{1}{2}\eta$, $-\frac{1}{2}\eta$, and dividing by the denominator, we obtain a value for G of the form

$$G = Q + \frac{1}{24}\left(\xi^2 \frac{d^2 Q}{d x^2} + \eta^2 \frac{d^2 Q}{d y^2}\right) + \frac{1}{960}\left(\xi^4 \frac{d^4 Q}{d x^4} + \eta^4 \frac{d^4 Q}{d y^4}\right) + \text{\&c.} \quad \text{[XX.]}$$

This is the general expression for the values of G. To find G_0, G_1, &c., we substitute the values of Q_0, Q_1, &c., respectively, taken from equation [XVI. A]. Thus from a comparison of [XVI. A] and [XVI. B] we see that $Q_0 = \cos a$; from Figure 5, we see that

$$\cos a = \frac{x}{z} = \frac{x}{\sqrt{x^2 + y^2}};$$

$$\frac{d^2}{d x^2}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = -\frac{3 y^2 x}{(x^2 + y^2)^{\frac{5}{2}}};$$

$$\frac{d^2}{d y^2}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \frac{2 x y^2 - x^3}{(x^2 + y^2)^{\frac{5}{2}}}.$$

Substituting in equation [XX.], we find

$$G_0 = \frac{x}{z}\left\{1 + \frac{1}{24}\,\frac{2 y^2 - x^2}{z^4}\,\xi^2 - \frac{1}{8}\,\frac{y^2}{z^4}\,\eta^2\right\};$$

or, since $G' = 2\pi\, G$,

$$G_0' = 2\pi \frac{x}{z}\left\{1 + \frac{1}{24}\,\frac{2 y^2 - x^2}{z^4}\,\xi^2 - \frac{1}{8}\,\frac{y^2}{z^4}\,\eta^2\right\}.$$

G_1', G_2', &c. are found in the same way.

$$G_1' = \frac{2\pi y^2}{z^3}\left\{1 + \frac{1}{24}\left(\frac{2}{y^2} - 15\,\frac{x^2}{z^4}\right)\xi^2 + \frac{1}{8}\left(\frac{4 x^2 - y^2}{z^4}\right)\eta^2\right\};$$

[XXI. A.]

$$G_2' = \frac{3\pi y^2 x}{z^5}\left\{1 + \frac{1}{24}\left(\frac{2}{y^2} - \frac{25}{z^2} + \frac{35 y^2}{z^4}\right)\xi^2 + \frac{5}{24}\left(\frac{4 x^2 - 3 y^2}{z^4}\right)\eta^2\right\};$$

$$G_3' = \frac{4\pi}{z^7}\left(y^2 x^2 - \frac{y^4}{4}\right) + \frac{\pi\xi^2}{24 z^{11}}\{z^4(8 x^2 - 12 y^2) + 35 x^2 y^2 (5 y^2 - 4 x^2)\}$$
$$+ \frac{\pi\eta^2}{24 z^{11}}\{3 y^2 z^2 (5 y^2 - 44 x^2) + 63 x^2 y^2 (4 x^2 - y^2)\}.$$

This same method of approximation may be applied to equation [XV.]:

$$\Omega' = 2\pi \frac{\sin^2 a}{2} \frac{a^2}{r^2} P_1(\theta) \frac{d P_1(a)}{d \cos a} + \text{\&c.}$$
$$+ 2\pi \sin^2 a \frac{a^{i+1}}{i+1} \frac{1}{r^{i+1}} \frac{d P_i(a)}{d \cos a} P_i(\theta). \quad \text{[XV.]}$$

Substituting q for those terms depending upon x and y,

$$\Omega' = 2\pi q_1 \frac{1}{r^2} P_1(\theta) + \text{\&c.} + 2\pi q_i \frac{1}{r^{i+1}} P_i(\theta), \quad \text{[XXI. B.]}$$

the value for one turn. Let g be the mean value for all the turns within the coil of cross-section $\xi\eta$, as deduced from the expression

$$g = q + \frac{1}{24}\left(\xi^2 \frac{d^2 q}{d x^2} + \eta^2 \frac{d^2 q}{d y^2}\right) + \text{\&c.}$$

Then, for the potential due to the whole coil,

$$\Omega' = 2\pi g_1 \frac{1}{r^2} P_1(\theta) + \text{\&c.} + 2\pi g_i \frac{1}{r^{i+1}} P_i(\theta);$$

or, calling $g_1' = 2\pi g_1$,

$$\Omega' = g_1' \frac{1}{r^2} P_1(\theta) + g_2' \frac{1}{r^3} P_2(\theta) + \text{\&c.}; \quad \text{[XXII.]}$$

$$q_1 = \frac{\sin^2 a}{2} a^2 \frac{d P_1(a)}{d \cos a} = \frac{1}{2} \frac{y^2}{z^2} z^2 = \tfrac{1}{2} y^2;$$

$$g_1 = \tfrac{1}{2} y^2 + \tfrac{1}{24} \xi^2;$$

$$g_1' = \pi y^2 + \tfrac{1}{12} \pi \xi^2; \quad \text{[XXIII.]}$$

$$g_2' = 2\pi y^2 x + \tfrac{1}{6} \pi x \xi^2;$$

$$g_3' = 3\pi y^2 (x^2 - \tfrac{1}{4} y^2) + \tfrac{\pi}{8} \xi^2 (2x^2 - 3y^2) + \tfrac{\pi}{4} \eta^2 y^2.$$

Equations [XIX.] and [XXII.] give the value of the potential at any point due to the current in a coil of any shape.

If now, instead of finding the potential at a point, we wish to find the force exerted by two coils, the one upon the other, we may replace the point by a coil and calculate the mutual potential of the two coils from the formulæ already deduced. Let us suppose, at first, that the two coils are co-axial. Then we may replace the coils by spherical magnetic shells, which are concentric. The radius of the larger shell will be a_1 and of the smaller a_2; a_1 and a_2 will be the angular radius of the larger and smaller coils respectively. Let Ω be the potential due to the first shell at any point within it, then the work required to carry the second shell to an infinite distance is given by the equation

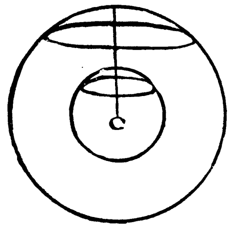

Fig. 6.

$$M = -\int\int \frac{d\,\Omega}{d\,r}\, d\,S$$

(see Maxwell, § 423), extended over the smaller shell. Hence, since $\int d\,S = 2\,\pi\,a_2^{\,2} \int_{\cos a_2}^{1} d\cos\theta$,

$$M = \int_{\cos a_2}^{1} \frac{d\,\Omega}{d\,r}\, 2\,\pi\,a_2^{\,2}\, d\cos\theta, \qquad \text{[XXIV.]}$$

θ being the varying angular radius of the smaller coil.

In equation [XIV.]

$$\Omega = -2\,\pi + 2\,\pi\cos a_1 - 2\,\pi\sin^2 a_1\,\frac{r}{a_1}\,\frac{d\,P_1(a_1)}{d\cos a_1}\,P_1(\theta) + \text{\&c.}$$
$$+\,2\,\pi\,\frac{\sin^2 a_1}{i}\,\frac{r^i}{a_1^{\,i}}\,\frac{d\,P_i(a_1)}{d\cos a_1}\,P_i(\theta)\,;$$

differentiating Ω with respect to r, and substituting in [XXIV.],

$$M = 4\,\pi^2\sin^2 a_1\,a_2^{\,2}\left[\frac{1}{a_1}\,\frac{d\,P_1(a_1)}{d\cos a_1}\int_{\cos a_2}^{1} P_1(\theta)\,d\cos\theta + \text{\&c.}\right.$$
$$\left. + \frac{r^{i-1}}{a_1^{\,i}}\,\frac{d\,P_i(a_1)}{d\cos a_1}\int_{\cos a_2}^{1} P_i(\theta)\,d\cos\theta\right].$$

Substituting the value of $\int_{\cos a_2}^{1} P(\theta)\,d\cos\theta$ from [IX. B] page 352, we find

$$M = 4\pi^2 \sin^2 a_1 \sin^2 a_2 a_2^2 \left[\frac{1}{2a_1} \frac{dP_1(a_1)}{d\cos a_1} \frac{dP_1(a_2)}{d\cos a_2} + \&c. \right.$$
$$\left. + \frac{1}{i(i+1)} \frac{r^{i-1}}{a_1^i} \frac{dP_i(a_1)}{d\cos a_1} \frac{dP_i(a_2)}{d\cos a_2}\right]. \qquad \text{[XXV.]}$$

If now the axes are inclined, making an angle θ with each other, we have as before (page 352) only to introduce the $P(\theta)$ of the corresponding order, and a_2 in place of r, which gives

$$M = 4\pi^2 \sin^2 a_1 \sin^2 a_2 a_2^2 \left[\frac{1}{2a_1} \frac{dP_1(a_1)}{d\cos a_1} \frac{dP_1(a_2)}{d\cos a_2} P_1(\theta) + \&c. \right.$$
$$\left. + \frac{1}{i(i+1)} \frac{a_2^{i-1}}{a_1^i} \frac{dP_i(a_1)}{d\cos a_1} \frac{dP_i(a_2)}{d\cos a_2} P_i(\theta)\right] \qquad \text{[XXVI.]}$$

as the expression for the mutual action of two circular currents of unit strength, the axes of the coils making an angle θ with each other at a point c distant a_1 and a_2 from the circumference.

It will be observed that equation [XXVI.] may be written

$$M = \left\{2\pi \sin^2 a_1 \frac{1}{a_1} \frac{dP_1(a_1)}{d\cos a_1}\right\} \left\{\pi \sin^2 a_2 a_2^2 \frac{dP_1(a_2)}{d\cos a_2}\right\} P_1(\theta) + \&c. +$$
$$\left\{2\pi \frac{\sin^2 a_1}{i} \frac{1}{a_1^i} \frac{dP_i(a_1)}{d\cos a_1}\right\} \left\{2\pi \sin^2 a_2 \frac{a_2^{i+1}}{i+1} \frac{dP_i(a_2)}{d\cos a_2}\right\} P_i(\theta).$$

The quantity within the first pair of brackets is precisely what we made Q_i' equal to in equation [XVI.], and the second quantity is what we made q_i' equal to in [XXI. B]; therefore equation [XXVI.] may be written

$$M = Q_1' q_1' P_1(\theta) + \&c. + Q_i' q_i' P_i(\theta). \qquad \text{[XXVII.]}$$

This is for one turn. If we have coils of a number of turns of wire whose sectional areas are $\xi\,\eta$, the Q' and q' will be replaced by the corresponding values of G' and g' as given in [XXI. A] and [XXIII.]. This gives the general expression for the mutual action of two coils in the form

$$M = G_1' g_1' P_1(\theta) + G_2' g_2' P_2(\theta) + \&c. + G_i' g_i' P_i(\theta). \qquad \text{[XXVIII.]}$$

If now one of the coils is free to move, so that there will be a variation in θ, the angle between the axes of the two coils, it is evident that the moment of the force tending to increase θ or F is $F = \frac{d\,M}{d\,\theta}$, or

$$F = G_1{}' g_1{}' \frac{d\,P_1(\theta)}{d\,\theta} + G_2{}' g_2{}' \frac{d\,P_2(\theta)}{d\,\theta} + G_3{}' g_3{}' \frac{d\,P_3(\theta)}{d\,\theta} + \&c.$$

$$F = - G_1{}' g_1{}' \sin\theta - G_2{}' g_2{}' \sin\theta \frac{d\,P_2(\theta)}{d\cos\theta} - G_3{}' g_3{}' \sin\theta \frac{d\,P_3(\theta)}{d\cos\theta} + \&c.$$

$$F = - \sin\theta\,(G_1{}' g_1{}' + G_2{}' g_2{}' \frac{d\,P_2(\theta)}{d\cos\theta} + \&c.).$$

Here θ is the angle between the axes of the coil; if we call θ the angle between the planes of the coils, we have

$$F = \cos\theta\,(G_1{}' g_1{}' + G_2{}' g_2{}' \frac{d\,P_2(\theta)}{d\cos\theta} + \&c.). \qquad [XXX.]$$

In all these equations G' is dependent upon the larger coil, and g' upon the smaller This same expression holds good if the smaller coil is replaced by a magnet, provided suitable values be given to g'. Let the magnet be long and uniformly magnetized. Let $2\,l$ be the length; then

$$g_1 = 2l; \qquad g_2 = 0; \qquad g_3 = -6\,l^3.$$

Take the case of a single coil galvanometer with a small magnetic needle; when the needle is at rest, the two couple are

$$F\,C\,m = H\,M \sin\theta;$$

substituting F from equation [XXX.], we have

$$C\,m\cos\theta\,(G_1{}' g_1{}' + G_2{}' g_2{}' \frac{d\,P_2(\theta)}{d\,\theta} + \&c.) = H\,M \sin\theta;$$

.or
$$C = \frac{H\,M}{m\,(G_1{}' g_1{}' + G_2{}' g_2{}' \frac{d\,P_2(\theta)}{d\,\theta} + \&c.)} \tan\theta;$$

or
$$C = \frac{H\,M}{m\,g_1'(G_1' + \frac{G_2'\,g_2'}{g_1'}\,\frac{d\,P_2(\theta)}{d\,\theta} + \&c.)}\tan\theta;$$

since $g_1' = 2\,l$, $m\,g_1' = M$, and

$$C = \frac{H}{G_1' + \frac{G_2'\,g_2'}{g_1}\,\frac{d\,P_2(\theta)}{d\,\theta} + \&c.}\tan\theta. \qquad \text{[XXXI.]}$$

If the needle is at the centre of the coil $z = y$, $x = 0$, and equations [XXI. A] become, if n equal the number of windings,

$$G_1' = \frac{2\,\pi\,n}{y}\left(1 + \frac{1}{12}\,\frac{\xi^2}{y^2} - \frac{5}{8}\,\frac{\eta^2}{y^2}\right);$$

$$G_2' = 0;$$

$$G_3' = -\frac{\pi\,n}{y^3}\left\{1 + \frac{1}{2}\,\frac{\xi^2}{y^2} - \frac{5}{8}\,\frac{\eta^2}{y^2}\right\};$$

$$G_4' = 0.$$

If we had but one turn, [XXXI.] would become

$$C = \frac{H\,y}{2\,\pi}\tan\theta,$$

the ordinary equation for a galvanometer.